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Lattice polytopes having h^*-polynomials with given degree and linear coefficient

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Abstract

The h^*-polynomial of a lattice polytope is the numerator of the generating function of the Ehrhart polynomial. Let P be a lattice polytope with h^*-polynomial of degree d and with linear coefficient h_1^*. We show that P has to be a lattice pyramid over a lower-dimensional lattice polytope if the dimension of P is greater than or equal to $h_1^*(2d + 1) + 4d - 1$. This result generalizes a recent theorem of Batyrev. As an application we deduce from an inequality due to Stanley that the volume of a lattice polytope is bounded by a function depending only on the degree and the two highest non-zero coefficients of the h^*-polynomial.

1. Introduction and main results

Let M be a lattice, and $P \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be an n-dimensional lattice polytope, i.e., the set of vertices of P, here denoted by $\mathcal{V}(P)$, is contained in the lattice M. Throughout, the normalized volume $\mathrm{Vol}(P)$ with respect to M is referred to as the volume of P. Moreover, two lattice polytopes $P \subseteq M_{\mathbb{R}}$ and $P' \subseteq M'_{\mathbb{R}}$ are called isomorphic if there is an affine lattice isomorphism $M \cong M'$ mapping $\mathcal{V}(P)$ onto $\mathcal{V}(P')$.

Due to Ehrhart and Stanley [4,10,11] the generating function enumerating the number of lattice points in multiples of P is a rational function of the following form:

$$\sum_{k \geq 0} |(kP) \cap M| \, t^k = \frac{h_0^* + h_1^* t + \cdots + h_n^* t^n}{(1-t)^{n+1}},$$

where $h_0^*, \ldots, h_n^*$ are non-negative integers satisfying the conditions $h_0^* = 1$, $h_1^* = |P \cap M| - n - 1$ and $h_0^* + \cdots + h_n^* = \mathrm{Vol}(P)$.

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Definition 1. The polynomial $h^*_P(t) := h^*_0 + h^*_1 t + \cdots + h^*_n t^n$ is called the *h^*-polynomial* of P (see [1,2,13]) or *δ-polynomial* (see [8]). The degree of $h^*_P(t)$, i.e., the maximal $i \in \{0, \ldots, n\}$ with $h^*_i \neq 0$, is called the *degree* $\deg(P)$ of P. We define the *codegree* of P as $\operatorname{codeg}(P) := n + 1 - \deg(P)$.

The geometric meaning of the codegree, introduced by Batyrev in [1], is given by the following observation:

$$\operatorname{codeg}(P) = \min(k \geq 1 \,:\, kP \text{ has interior lattice points}).$$

The notion of the degree of a lattice polytope was defined in [2], where it was noted that $\deg(P)$ should be considered the "lattice dimension" of P. This interpretation of the degree was motivated by the following three basic properties: First, $\deg(P) = 0$ if and only if $\operatorname{Vol}(P) = 1$. So the unimodular simplex is the only lattice polytope with degree zero. Second, by Stanley's monotonicity theorem [13], $h^*_Q(t) \leq h^*_P(t)$ holds coefficientwise for lattice polytopes $Q \subseteq P$. In particular this implies that the degree is monotone with respect to inclusion. For the third property let us recall the notion of lattice pyramids [1]:

Definition 2. Let $B \subseteq \mathbb{R}^k$ be a lattice polytope with respect to $\mathbb{Z}^k$. Then $\operatorname{conv}(0, B \times \{1\}) \subseteq \mathbb{R}^{k+1}$ is a lattice polytope with respect to $\mathbb{Z}^{k+1}$, called the (1-fold) standard pyramid over B. Recursively, we define for $l \in \mathbb{N}_{\geq 1}$ in this way the l-fold standard pyramid over B. As a convention, the 0-fold standard pyramid over B is B itself. Now, let $P, Q \subseteq M_{\mathbb{R}}$ be lattice polytopes with $Q \subseteq P$. We say P is a *lattice pyramid* over Q, if $P \subseteq M_{\mathbb{R}}$ is isomorphic to the $(\dim(P) - \dim(Q))$-fold standard pyramid over a lattice polytope B, where this isomorphism maps Q onto B.

Now, for lattice polytopes $Q \subseteq P$ we observe that P is a lattice pyramid over Q if and only if $\operatorname{Vol}(P) = \operatorname{Vol}(Q)$, or equivalently, $h^*_P(t) = h^*_Q(t)$ (e.g., see [1]). This implies as a third property the invariance of the degree under lattice pyramid constructions.

In [1] Batyrev showed the following theorem:

Theorem 3 (*Batyrev*). *Let $P \subseteq M_{\mathbb{R}}$ be an n-dimensional lattice polytope of volume V and degree d. If*

$$n \geq 4d \binom{2d + V - 1}{2d},$$

then P is a lattice pyramid over an $(n-1)$-dimensional lattice polytope.

Recursively, we see that any lattice polytope P is a lattice pyramid over a lattice polytope Q with $h^*_P(t) = h^*_Q(t)$, where the dimension of Q is bounded by a function depending only on the degree and the volume of P. Since by [9] there are up to isomorphisms only a finite number of n-dimensional lattice polytopes with volume V, if n and V are fixed, we get the following corollary:

Corollary 4 (*Batyrev*). *There are only a finite number of lattice polytopes of fixed degree d and fixed volume V up to isomorphisms and lattice pyramid constructions.*

Here, we improve the bound in Batyrev's theorem to the presumably correct asymptotic behaviour:

Theorem 5. *Let $P \subseteq M_{\mathbb{R}}$ as in Theorem 3. If*

$$n \geq (V-1)(2d+1),$$

then P is a lattice pyramid over an $(n-1)$-dimensional lattice polytope.

Note that for $d = 1$ this yields the assumption $n \geq 3(V-1)$, while Batyrev's theorem needs $n \geq 2(V+1)V$. Since all lattice polytopes of degree 1 were classified in [2], it could be observed in [1, Prop. 4.1] that $n \geq V+1$ is the optimal bound.

Example 6. Here is an example of a lattice polytope with degree $d \geq 2$, volume $V = 2$, and dimension $n = 2d-1$ that is not a lattice pyramid: the simplex with vertices $e_0 - e_n, e_1 - e_n, \ldots, e_{n-1} - e_n, e_0 + \cdots + e_{n-1} + (3-2d)e_n$, where $e_0, \ldots, e_n$ is a lattice basis of $\mathbb{Z}^{n+1}$. The h^*-polynomial equals $1 + t^d$. Though this example does not show that the bound given in Theorem 5 is sharp, we see again that the asymptotics seems to have the right order.

While Batyrev's proof involved commutative and homological algebra, our methods are elementary and purely combinatorial.

The main result of this paper is a generalization of Batyrev's theorem. We show that the qualitative statement of Theorem 3 still holds when we replace the volume of P by the "relative" number of vertices $|\mathcal{V}(P)| - n - 1$, which is in contrast to the volume being an invariant that depends only on the combinatorics of P.

Theorem 7. *Let $c, d \in \mathbb{N}$. Let $P \subseteq M_{\mathbb{R}}$ be an n-dimensional lattice polytope having $\leq c+n+1$ vertices and degree $\leq d$. If*

$$n \geq c(2d+1) + 4d - 1,$$

then P is a lattice pyramid over an $(n-1)$-dimensional lattice polytope.

Note that for $d = 1$ this yields the assumption $n \geq 3(c+1)$, while the optimal bound is $n \geq 3$ for $c = 0$ and $n \geq c+2$ for $c > 0$ by the classification [2].

Now, since $|\mathcal{V}(P)| - n - 1 \leq |P \cap M| - n - 1 = h_1^*$, we see that the implication in Theorem 7 holds for $c = h_1^*$. This result motivates the following more general conjecture:

Conjecture 8. *Let c, d, and $i \in \{1, \ldots, d\}$ be fixed. Then there is a function f_i depending only on c and d such that any n-dimensional lattice polytope P with $h_i^* = c$ and degree* $\deg(P) = d$ *is a lattice pyramid over an $(n-1)$-dimensional lattice polytope if $n \geq f_i(c, d)$.*

For $i = 1$, the conjecture holds, as we have just seen. For $i = 2, \ldots, d-1$, the conjecture would follow from the inequalities $h_1^* \leq h_i^*$. In the case of $d = n$, these were proven by Hibi [8]. The author is not aware of any counterexamples for arbitrary degree.[1]

For $i = d$, Batyrev's theorem implies that Conjecture 8 is equivalent to Conjecture 4.2 in [1], saying that $\mathrm{Vol}(P)$ should be bounded by a function in d and h_d^*. This equivalence follows from the invariance of the h^*-polynomial under lattice pyramid constructions, the fact [1] that $h_d^* > 0$ equals the number of interior lattice points in $\mathrm{codeg}(P)P$, and a result due to Hensley [7] that the volume of any n-dimensional lattice polytope with $l > 0$ interior lattice points is bounded by a function depending only on n and l.

[1] In the meantime Henk and Tagami provided a counterexample [6].

By the same argument we deduce from Theorem 7 a generalization of Corollary 4 using the finiteness result of Lagarias and Ziegler [9], already mentioned before Corollary 4, and the following inequality due to Stanley [12, Prop. 4.1]:

$$1 + h_1^* \leq h_{d-1}^* + h_d^*.$$

Corollary 9. *Let $i \in \{1, d-1\}$. There are only a finite number of lattice polytopes of fixed degree d and with fixed h_i^* and h_d^* up to isomorphisms and lattice pyramid constructions. In particular, the volume of any lattice polytope of degree d is bounded by a function depending only on d, h_i^* and h_d^*.*

The paper is organized as follows. In the second section we deal with lattice simplices of degree d, showing that they are lattice pyramids over lower-dimensional lattice simplices if their dimension is larger than $4d - 2$. On the basis of this result we prove in the third section Theorems 5 and 7.

2. Lattice simplices with fixed degree

In this section we prove Theorem 7 for $c = 0$:

Theorem 10. *Any lattice simplex of degree $\leq d$ and dimension $n \geq 4d - 1$ is a lattice pyramid over an $(n - 1)$-dimensional lattice simplex.*

The bound $4d - 1$ is sharp for $d \leq 1$; see [2].

Through the whole section let $M = \mathbb{Z}^{n+1}$, and $P = \operatorname{conv}(v_0, \ldots, v_n)$ be an n-dimensional lattice simplex of degree d, embedded in $M_{\mathbb{R}} = \mathbb{R}^{n+1}$ on the affine hyperplane $\mathbb{R}^n \times \{1\}$. We define the half-open *parallelepiped*

$$\Pi(P) := \left\{ \sum_{i=0}^{n} \lambda_i v_i \; : \; \lambda_i \in [0, 1[\right\}.$$

Moreover, for $x = \sum_{i=0}^{n} \lambda_i v_i \in \Pi(P) \cap M$ we define its *support*

$$\operatorname{supp}(x) := \{i \in \{0, \ldots, n\} : \lambda_i \neq 0\}$$

and its *height* as the last coordinate of x:

$$\operatorname{ht}(x) := \sum_{i=0}^{n} \lambda_i \in \mathbb{N}.$$

It is well known [3, Cor. 3.11] that h_i^* equals the number of lattice points in $\Pi(P)$ of height i. From this observation, we derive the following result:

Lemma 11. *Let $m \in \Pi(P) \cap M$. Then $|\operatorname{supp}(m)| \leq 2d$.*

Proof. Let $m = \sum_{i=0}^{s} \lambda_i v_i$ with $\lambda_i \neq 0$ for $i = 0, \ldots, s$. We define $P' := \operatorname{conv}(v_0, \ldots, v_s)$. Then m is a lattice point in the relative interior of $\operatorname{ht}(m) \cdot P'$, so $s + 1 - \deg(P') = \operatorname{codeg}(P') \leq \operatorname{ht}(m)$; hence $s + 1 \leq \operatorname{ht}(m) + \deg(P')$. Since $m \in \Pi(P) \cap M$, we have $\operatorname{ht}(m) \leq d$, and by monotonicity $\deg(P') \leq d$. Therefore $|\operatorname{supp}(m)| = s + 1 \leq 2d$. $\square$

Let us define the *support* of P as

$$\operatorname{supp}(P) := \bigcup_{m \in \Pi(P) \cap M} \operatorname{supp}(m) \subseteq \{0, \dots, n\}.$$

The relation of this notion to lattice pyramids is straightforward:

Lemma 12. *Let $i \in \{0, \dots, n\}$. Then P is a lattice pyramid with apex v_i if and only if $i \notin \operatorname{supp}(P)$.*

Proof. Let $P' := \operatorname{conv}(v_j : j = 0, \dots, n,\ j \neq i)$. Then P is a lattice pyramid over P' if and only if $\operatorname{Vol}(P) = \operatorname{Vol}(P')$. Now, the statement follows from $\operatorname{Vol}(P) = |\Pi(P) \cap M| \geq |\Pi(P') \cap M| = \operatorname{Vol}(P')$. $\square$

Now, we can give the proof of Theorem 10:

Proof of Theorem 10. By Lemma 12 it is enough to show

$$|\operatorname{supp}(P)| \leq 4d - 1.$$

Let $m_0 \in \Pi(P) \cap M$ with $I_0 := \operatorname{supp}(m_0)$ maximal. Now, we choose successively in a "greedy" manner lattice points $m_0, m_1, \dots, m_k \in \Pi(P) \cap M$ such that $|I_k|$ is maximal, where

$$I_k := \operatorname{supp}(m_k) \setminus \left(\bigcup_{j=0}^{k-1} \operatorname{supp}(m_j) \right).$$

Claim: For $k \in \mathbb{N}$ we have $|I_k| \leq \frac{2d}{2^k}$.

Assume the claim to be already proven. Then, since $|\Pi(P) \cap M| = \operatorname{Vol}(P)$ is finite, the construction yields

$$|\operatorname{supp}(P)| = \left| \bigcup_{k=0}^{|\Pi(P) \cap M|} \operatorname{supp}(m_k) \right| < \sum_{k=0}^{\infty} \frac{2d}{2^k} = 4d.$$

This proves the theorem. It remains to show the claim:

The claim holds for $k = 0$ by Lemma 11. Let it be true for $k - 1 \in \mathbb{N}$. We set $J_k := I_{k-1} \cap \operatorname{supp}(m_k)$. This implies

$$J_k \sqcup I_k \subseteq \operatorname{supp}(m_k) \setminus \left(\bigcup_{j=0}^{k-2} \operatorname{supp}(m_j) \right).$$

Hence, by the choice of m_{k-1} with I_{k-1} maximal we get

$$|J_k| + |I_k| \leq |I_{k-1}|. \tag{1}$$

On the other hand, let $m_{k-1} = \sum_{i=0}^{n} \lambda_i v_i$ and $m_k = \sum_{i=0}^{n} \mu_i v_i$. Now, we translate $m_{k-1} + m_k$ into $\Pi(P)$:

$$m := \sum_{i=0}^{n} \{\lambda_i + \mu_i\} v_i \in \Pi(P) \cap M,$$

where $\{\gamma\} \in [0, 1[$ denotes the fractional part of $\gamma \in \mathbb{R}$. By construction, $\mu_i = 0$ and $\{\lambda_i + \mu_i\} = \lambda_i > 0$ for $i \in I_{k-1} \setminus J_k$, as well as $\lambda_i = 0$ and $\{\lambda_i + \mu_i\} = \mu_i > 0$ for

$i \in I_k$. This implies

$$(I_{k-1} \setminus J_k) \sqcup I_k \subseteq \mathrm{supp}(m) \setminus \left(\bigcup_{j=0}^{k-2} \mathrm{supp}(m_j) \right).$$

Again, by the maximality of $|I_{k-1}|$ we get

$$|I_{k-1}| - |J_k| + |I_k| \leq |I_{k-1}|. \tag{2}$$

Combining Eqs. (1) and (2) yields

$$|I_k| \leq |J_k| \leq |I_{k-1}| - |I_k|.$$

Hence, $|I_k| \leq |I_{k-1}|/2 \leq \frac{2d}{2^k}$ by induction hypothesis. This proves the claim. $\square$

3. Proof of Theorems 5 and 7

Throughout, let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope of dimension n and degree $\leq d$. The proofs here are based on induction. For the induction step we need the notion of a circuit:

Definition 13. An affinely dependent subset $\mathcal{C} \subseteq \mathcal{V}(P)$ is called a *circuit* in P if any proper subset of $\mathcal{C}$ is affinely independent.

The importance of this notion lies in the fact that P is combinatorially a pyramid with apex $v \in \mathcal{V}(P)$ if and only if v is not contained in any circuit in P.

The following observation [5, Lemma 2.1] is joint work with Christian Haase and Andreas Paffenholz:

Lemma 14. *Any circuit in P consists of $\leq 2d + 2$ elements.*

Proof. Let $\mathcal{C}$ be a circuit in P. We may assume as in the previous section that P is embedded in $\mathbb{R}^{n+1}$ on the affine hyperplane with last coordinate 1. In this case, there is a linear relation among the elements of $\mathcal{C}$, i.e.,

$$\sum_{v \in \mathcal{C}_1} z_v v = \sum_{w \in \mathcal{C}_2} z_w w$$

for $\mathcal{C} = \mathcal{C}_1 \sqcup \mathcal{C}_2$ and $z_v, z_w \in \mathbb{N}_{>0}$. Let $Q := \mathrm{conv}(\mathcal{C})$. The dimension of Q equals $|\mathcal{C}_1| + |\mathcal{C}_2| - 2$. We observe that $\sum_{v \in \mathcal{C}_1} v$ is a lattice point in the relative interior of $|\mathcal{C}_1| \cdot Q$. Thus, $\mathrm{codeg}(Q) \leq |\mathcal{C}_1|$, so by monotonicity $d \geq \deg(Q) = \dim(Q) + 1 - \mathrm{codeg}(Q) \geq |\mathcal{C}_2| - 1$. Hence $|\mathcal{C}_2| \leq d + 1$. Symmetrically, $|\mathcal{C}_1| \leq d + 1$. This proves the statement. $\square$

Using this lemma we can prove Theorem 7.

Proof of Theorem 7. First, let us define $n(c, d) := c(2d + 1) + 4d - 1$. Now, we prove by induction on $c \geq 0$ that any n-dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ having $\leq c + n + 1$ vertices and degree $\leq d$ is a lattice pyramid over a lattice polytope of dimension $< n(c, d)$.

So, let P be given in this way, and $n \geq n(c, d)$.

If $c = 0$, then $|\mathcal{V}(P)| = n + 1$, so P is a simplex, and the statement follows from Theorem 10, since $n(0, d) = 4d - 1$.

Let $c \geq 1$. Since P is not a simplex, there is a vertex $v \in \mathcal{V}(P)$ such that $Q := \mathrm{conv}(\mathcal{V}(P) \setminus \{v\})$ is an n-dimensional lattice polytope. Since $(|\mathcal{V}(Q)| - n - 1) < (|\mathcal{V}(P)| - n - 1) \leq c$, the induction hypothesis yields that Q is a lattice pyramid over a lattice polytope B with $\dim(B) < n(c-1,d)$.

Now, since $\dim(Q) = \dim(P)$, there is a circuit in P containing vertices $v, w_1, \ldots, w_l$, where $w_j \in \mathcal{V}(Q)$ (for $j = 1, \ldots, l$), and $l \leq 2d+1$ by Lemma 14. In particular, $v \in \operatorname{aff}(w_1, \ldots, w_l)$. We set $D := \operatorname{conv}(B, w_1, \ldots, w_l) \subseteq Q$. Hence, Q is a lattice pyramid over the lattice polytope D, whose dimension satisfies

$$\dim(D) \leq \dim(B) + l \leq (n(c-1,d) - 1) + (2d+1) = n(c,d) - 1.$$

Since $\operatorname{aff}(D) = \operatorname{aff}(D, v)$, also P is a lattice pyramid over the lattice polytope $\operatorname{conv}(D, v)$ of dimension $\dim(D) < n(c,d)$. $\square$

The proof of Theorem 5 is analogous.

Proof of Theorem 5. The proof is by induction on $V \geq 1$.

Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope having volume V, degree d, and $\dim(P) = n \geq (V-1)(2d+1)$. If $V = 1$, the statement is trivial. So, let $V \geq 2$.

First, let P be a lattice simplex. If $V \geq 3$, then $n \geq 4d+2$, so the statement follows from Theorem 10. If $V = 2$, then there exists in the notation of the previous section precisely one lattice point $0 \neq m \in \Pi(P) \cap M$. Hence, $|\operatorname{supp}(P)| = |\operatorname{supp}(m)| \leq 2d$ by Lemma 11, so P is a lattice pyramid over an $(n-1)$-dimensional lattice simplex by Lemma 12, since $n \geq 2d+1$.

Therefore, we can assume that P is not a simplex. Now, the remaining induction step proceeds precisely as in the proof of Theorem 7. $\square$

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